## Comment on "Hidden quantum nonlocality revealed by local filters"

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In Section 3 of his paper [1], Gisin argues that a "careless application of generalized quantum measurements can violate Bell's inequality even for mixtures of product states." However, the observed violation of the CHSH inequality is not in fact due to the application of generalized quantum measurements, but rather to a misapplication of the inequality itself — to conditional expectations in which the conditioning depends upon the measurements under consideration.

Consider the usual setup of quantum nonlocality arguments: a system consists of two widely separated subsystems, and in each of the subsystems one of two possible experiments a, a' and b, b', respectively, can be performed. The system is in a quantum state  $\rho$  which is a density matrix on the Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . We shall denote by  $O_a^i$  the positive operator on  $\mathcal{H}_1$  giving via  $\operatorname{tr} \rho (O_a^i \otimes I)$  the probability of obtaining the value i for the measurement of a. Similarly  $O_b^j$  denotes the positive operator on  $\mathcal{H}_2$  for the value j of the measurement of b.

A local hidden variables model for this setup consists of random variables  $X_a$ ,  $X_{a'}$ ,  $X_b$ , and  $X_{b'}$  for the experiments under consideration on some probability space  $(\Omega, \mathbb{P})$  such that the joint distributions of the model reproduce the quantum joint distribution  $P^{\rho}$ 

$$\mathbb{P}(X_a = i, X_b = j) = P^{\rho}(a = i, b = j) = \operatorname{tr} \rho \left( O_a^i \otimes O_b^j \right) \tag{1}$$

for a joint measurement of a and b, and similarly for the pairs (a, b'), (a', b), and (a', b'). Thus we have for the expectation value of the product  $a \cdot b$ 

$$E^{\rho}(a \cdot b) = \mathbb{E}(X_a X_b) \left( = \int_{\Omega} X_a(\omega) X_b(\omega) d\mathbb{P}(\omega) \right), \tag{2}$$

<sup>&</sup>lt;sup>1</sup>This covers measurements associated with positive operator valued (POV) measures, as well as the special case of measurements associated with self-adjoint operators, where  $O_a^i$  will be the projection on the eigenspace of the eigenvalue i.

and similarly for the pairs (a,b'), (a',b), and (a',b'). Random variables  $X_n$ , n=1...4 taking values in [-1,1] satisfy the CHSH inequality [2,3]

$$\mathbb{E}(X_1 X_2) + \mathbb{E}(X_1 X_3) + \mathbb{E}(X_4 X_2) - \mathbb{E}(X_4 X_3) \le 2. \tag{3}$$

Thus (2) implies that if there is a local hidden variables model for quantum measurements taking values in [-1, 1] then

$$E^{\rho}(a \cdot b) + E^{\rho}(a \cdot b') + E^{\rho}(a' \cdot b) - E^{\rho}(a' \cdot b') \le 2, \tag{4}$$

and the violation of (4) proves that a local hidden variables model for the considered setup is impossible.

A quantum state  $\rho$  that is a product state quite obviously allows for a local hidden variables model reproducing the distributions of local experiments, and thus this must also be true of a mixture of product states. Moreover, this conclusion holds regardless of the nature of the local experiments and in particular it does not matter whether these experiments are described by standard observables represented by self-adjoint operators or by generalized observables represented by positive operator valued (POV) measures.<sup>2</sup> Thus for quantum measurements with results in [-1, 1], (4) must be satisfied in a mixture of product states.

Nevertheless, Gisin presents POV's — which may be regarded as corresponding to the three possible outcomes -1, 0, 1 — which apparently yield a violation of (4) even for a state which is a mixture of product states. While the expectations that he considers only concern the instances in which the particles both first pass through a filter, one would expect the ensemble so defined to still be local and hence to still satisfy (4), even for generalized observables.

We thus must more carefully analyze the expectation values used by Gisin. The expectation value of the product of a and b — our a corresponds to Gisin's  $(\alpha, \mathbf{a})$ , b to  $(\beta, \mathbf{b})$  etc. — is given by

$$E^{\rho}(a \cdot b) = \sum_{i,j \in \{-1,0,1\}} ij \ P^{\rho}(a=i,b=j)$$

$$= P^{\rho}(a=1,b=1) + P^{\rho}(a=-1,b=-1)$$

$$-P^{\rho}(a=1,b=-1) - P^{\rho}(a=-1,b=1)$$
(5)

and analogously for  $E^{\rho}(a \cdot b')$ ,  $E^{\rho}(a' \cdot b)$ , and  $E^{\rho}(a' \cdot b')$ . This equals the numerator in Gisin's Eqn. (10), and this quantity cannot violate (4) when calculated in a quantum state which is a mixture of product states. But the quantities for which Gisin shows that they can lead to a violation of (4) — Gisin's Eqn. (10) — are not  $E^{\rho}(a \cdot b)$  but the conditional expectation values

$$E^{\rho}(a \cdot b | a \neq 0, b \neq 0) = \sum_{i,j \in \{-1,0,1\}} ij \ P^{\rho}(a = i, b = j | a \neq 0, b \neq 0)$$
$$= \frac{E^{\rho}(a \cdot b)}{P^{\rho}(a \neq 0, b \neq 0)}$$
(6)

<sup>&</sup>lt;sup>2</sup>Gisin also states this in the second sentence after Eqn. (12) in [1]. We have proven a more general statement in [4].

conditioned under "both outcomes different from zero," an event that depends upon the the choice of experiments. In a local hidden variables model these conditional expectations are represented by

$$E^{\rho}(a \cdot b | a \neq 0, b \neq 0) = \mathbb{E}(X_a X_b | X_a \neq 0, X_b \neq 0) = \frac{\mathbb{E}(X_a X_b)}{\mathbb{P}(X_a \neq 0, X_b \neq 0)}.$$
 (7)

Clearly, since for the different random variables  $X_a$ ,  $X_{a'}$ ,  $X_b$ , and  $X_{b'}$  these conditional expectations refer to different subensembles of the original ensemble defined by  $(\Omega, \mathbb{P})$ , in general the conditional expectations need not satisfy (3), and thus the violation of this inequality does not preclude the existence of a local hidden variables model in this case. In Gisin's example the subensembles  $(X_a \neq 0, X_b \neq 0)$ ,  $(X_a \neq 0, X_{b'} \neq 0)$ , etc., correspond to the event that the photons pass the filters which are put in the directions  $(\mathbf{a}, \mathbf{b})$ ,  $(\mathbf{a}, \mathbf{b}')$ , etc., respectively, and the problem arises simply — as Gisin points out — because the "filters depend on the measured quantity" which amounts to selecting experiment-dependent subensembles.

Thus the results Gisin presents in Section 3 of [1] are not at all "bizarre." Nor are they related to the application of POV measures. In fact, one can easily construct a similar situation with 3-valued standard observables (however, of course not in Hilbert space dimension 2): take 2 spin-1 particles,  $\mathcal{H} = \mathbb{C}^3 \times \mathbb{C}^3$ , consider the spin observables  $J_{\alpha}$  in direction  $\alpha$  in the x-z-plane

$$J_{\alpha} = \begin{pmatrix} \cos \alpha & \frac{\sin \alpha}{\sqrt{2}} & 0\\ \frac{\sin \alpha}{\sqrt{2}} & 0 & \frac{\sin \alpha}{\sqrt{2}}\\ 0 & \frac{\sin \alpha}{\sqrt{2}} & -\cos \alpha \end{pmatrix},$$

and take the state  $\rho = \frac{1}{2} P_{(1,0,0)\otimes(1,0,0)} + \frac{1}{2} P_{(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2})}$ . Then the conditional expectations  $\tilde{E}^{\rho}(\alpha,\beta) = E^{\rho}(J_{\alpha}\otimes J_{\beta}|J_{\alpha}\neq 0, J_{\beta}\neq 0)$  satisfy  $\tilde{E}^{\rho}(\alpha,\beta) + \tilde{E}^{\rho}(\alpha',\beta) + \tilde{E}^{\rho}(\alpha',\beta') - \tilde{E}^{\rho}(\alpha',\beta') = \frac{16}{9}\sqrt{2} > 2$  for the choice  $\alpha = 0, \beta = \frac{\pi}{4}, \alpha' = \frac{\pi}{2}, \beta' = -\frac{\pi}{4}$ . This, however, as explained above, tells nothing about nonlocality, i.e., the nonexistence of a local hidden variables model.

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## References

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